IHES lecture 2

Sunday, July 4, 2021 6:53 AM

Braid varieties

$$B_i(z) = \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & 1 & z & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix}$$ braid matrices

<u>Exercise</u> $B_i(z_1)B_{i+1}(z_2)B_i(z_3) =$

$(*) \qquad B_{i+1}(z_3)B_i(z_2 - z_1 z_3)B_{i+1}(z_1)$

$\beta = \sigma_{i_1} \ldots \sigma_{i_r} \rightsquigarrow B_\beta(z_1 \ldots z_r) = B_{i_1}(z_1) \cdots B_{i_r}(z_r)$

$$\boxed{X(\beta) = \{ z_1 \ldots z_r \mid B_\beta(z_1 \ldots z_r) \text{ is upper-triangular} \}}$$

This is an explicit affine algebraic variety in $\mathbb{C}^r$, by $(*)$ varieties for equivalent braids are isomorphic.

<u>Ex:</u> $\beta = \sigma_1^4$

$$\begin{pmatrix} 0 & 1 \\ 1 & z_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & z_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & z_3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & z_4 \end{pmatrix} = \begin{pmatrix} * & * \\ z_1 + z_3 + z_1 z_2 z_3 & * \end{pmatrix}$$

$X(\beta) = \{ z_1 + z_3 + z_1 z_2 z_3 = 0 \}$ does not depend on z_4

$\int$

z_4

$$z_1 + z_3(1+z_1z_2) = 0$$

$1+z_1z_2 = 0$

$z_1 = 0$

contradiction

$1+z_1z_2 \neq 0$

$$z_3 = -\frac{z_1}{1+z_1z_2}.$$

open in $\mathbb{C}^2$

<u>Conclusion</u> $X(\beta) = \{1+z_1z_2 \neq 0\} \times \mathbb{C}_{z_4}$

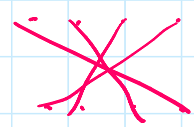
smooth, dim $= 3$, noncompact.

<u>Note:</u> $\{1+z_1z_2 \neq 0\}$ is a cluster variety of type A_1 with one frozen variable.

There is a torus action $(z_1, z_2) \to (tz_1, t^{-1}z_2)$.

<u>Thm</u> (Casals, G., M. Gorsky, Simental)

Suppose that $\beta = \gamma\Delta$ ← half-twist



Then:

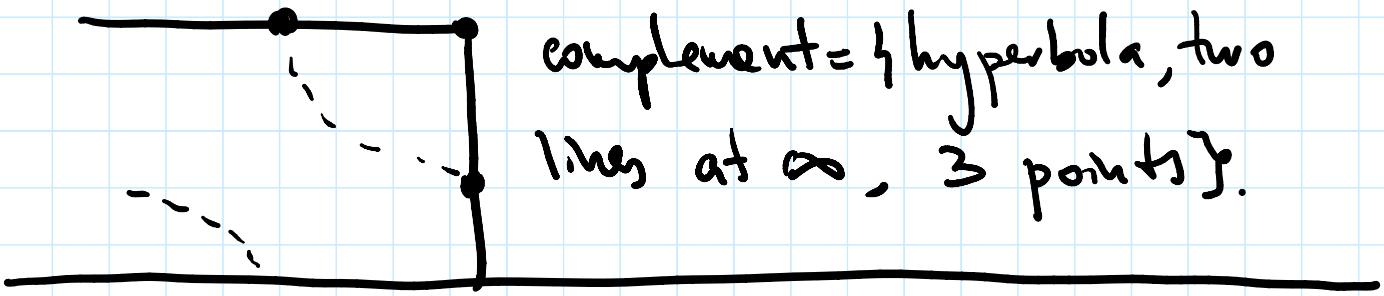
(a) $X(\beta)$ is non-empty iff γ contains Δ as a subword. In this case it is smooth of (expected) dim $= \ell(\beta) - \binom{n}{2} = \ell(\gamma)$.

(b) $X(\beta)$ is an invariant of

(b) $X(\beta)$ is an invariant of $\gamma\Delta^{-1}$ up to $\mathbb{C}^{\cdots}\times(\mathbb{C}^*)^{\cdots}$ under conjugation and positive stabilization.

(c) $X(\beta)$ has a smooth compactification (depending on a braid word for β), complement = normal crossing divisor, components $\leftrightarrow$ subwords of γ containing Δ.

<u>Ex</u> $\{1+z_1z_2\neq 0\}$ compactifies to $\mathbb{P}^1\times\mathbb{P}^1$



complement = {hyperbola, two lines at ∞, 3 points}.

<u>Thm2</u> (Webster-Williamson, Mellit...) $X(\beta)\circlearrowleft T=(\mathbb{C}^*)^{n-1}$

The (equivariant) cohomology of $X(\beta)$ has a nontrivial weight filtration:

$$\mathrm{gr}_w H_T^{\cdot}(X(\beta)) = \text{(up to some regrading)}$$

$$= HHH^n(\beta) \leftarrow \text{"top KhR link homology"}$$

$$= HHH^0(\beta\Delta^{-2}) = HHH^0_a(\gamma\Delta^{-1})$$

$$= HHH^{0}(\beta\Delta^{-2}) = HHH^{0}(\gamma\Delta^{-1})$$

"bottom KhR link homology"

<u>Rmk</u> As we discussed in Lecture 1, HHH^0 is invariant under conjugation and positive stabilization.

<u>Rmk</u> $X(\beta)$ is paved by strata $\mathbb{C}^{\cdots} \times (\mathbb{C}^*)^{\cdots}$ so Hodge filtration is easy.

<u>Rmk</u> Very recently, M.-T. Trinh found a way to compute all HH^i using similar varieties and Springer theory.

<u>Idea of proof of Thm 2:</u>

Recall $B_i = R \otimes_{R^{s_i}} R$

<u>Bott-Samelson variety</u>

$$BS_i = \{(\mathcal{F}, \mathcal{F}') : \mathcal{F}_j = \mathcal{F}'_j \text{ for } j \neq i\}$$

complete flags

$$\mathcal{L}_j = \mathcal{F}_j / \mathcal{F}_{j-1} \qquad \mathcal{L}'_j = \mathcal{F}'_j / \mathcal{F}'_{j-1} \quad \text{line bundles}$$

$$x_i = c_1(\mathcal{L}_i) \qquad x'_i = c_1(\mathcal{L}'_i)$$

$x_j = c_1(\mathcal{L}_j) \qquad x'_j = c_1(\mathcal{L}'_j)$

$\mathcal{L}_j = \mathcal{L}'_j$ for $j \neq i, i+1$

$\mathcal{F}_{i+1}/\mathcal{F}_{i-1}$ is filtered both by $\mathcal{L}_i, \mathcal{L}_{i+1}$ and $\mathcal{L}'_i, \mathcal{L}'_{i+1}$

so symmetric functions in x_i, x_{i+1} and x'_i, x'_{i+1} agree.

$B_{i_1} \otimes \cdots \otimes B_{i_r} \rightsquigarrow (\mathcal{F}^{(1)}, \mathcal{F}^{(2)}, \ldots, \mathcal{F}^{(r+1)})$
sequence of flags

neighbors $\longrightarrow$ BS condition

$T_i = [B_i \to R] \rightsquigarrow$ open BS variety

$\mathcal{F}, \mathcal{F}'$ are in position s_i if

- $\mathcal{F}_j = \mathcal{F}'_j$ for $j \neq i$ (usual BS)
- $\mathcal{F}_i \neq \mathcal{F}'_i$ (remove diagonal)

$T_{i_1} \cdots T_{i_r} \longleftrightarrow (\mathcal{F}^{(1)}, \mathcal{F}^{(2)}, \ldots, \mathcal{F}^{(r+1)}) = OBS_p$

$\boxed{\text{Considered by Broué-Michel, Deligne, ...}}$

in position s_{i_1}, s_{i_2} ......

The complex of B's corresponding to

$T_{i_1} \cdots T_{i_r} \longleftrightarrow$ inclusion-exclusion formula for open BS

$\Gamma_{i_1}, \dots, \Gamma_{i_r} \longleftrightarrow$ inclusion-exclusion formula for open BS inside closed BS and boundary strata.

<u>Lemma</u> $X(\beta)$ = subset of OBS_β where both $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(r+1)}$ are <u>standard</u> flags.

Embedding $OBS_\beta \hookrightarrow BS_\beta$ corresponds to the compactification of $X(\beta)$ above.

<u>More examples:</u>

① (Galashin-Lam) Torus links correspond to open positroid strata in $Gr(k,n)$:

$$k\left\{\underbrace{\begin{pmatrix} | & & & | \\ v_1 & \cdots & \cdots & v_n \\ | & & & | \end{pmatrix}}_{n}\right.$$

Repeat periodically: $v_{i+n} = v_i$

$$\Pi_{k,n} = \{\det(v_i, v_{i+1} \dots v_{i+k-1}) \neq 0\} \Big/ \text{row operations}$$

$$\Delta_{i \dots i+n-1} \qquad / \text{ row operations}$$

$$X(T(k,n)) \rightsquigarrow \Pi_{k,n} \quad \text{up to some } \mathbb{C}^{\cdot\cdot} \times (\mathbb{C}^*)^{\cdots}$$

<u>Ex</u> $\Pi_{2,4} = \{1+z_1 z_2 \neq 0\} \times (\mathbb{C}^*)^2$

$$\Pi_{2,4} \times \mathbb{C} = X(\sigma^4) \times (\mathbb{C}^*)^2.$$

$\sigma^4 \leftrightarrow T(2,4).$

(2) $w, u \in S_n$, $w \geq u$ in Bruhat order

$X(\beta(w)\,\beta(u^{-1}w_0)\,\Delta) =$ open Richardson variety for w, u

positive braid lifts

(3) $w, u \in S_n$, $w \geq u$ and w is k-Grassmannian

$\rightsquigarrow$ more general positroid variety

$\Pi_{w,u} \subset Gr(k,n)$ (Knutson, Lam, Speyer)

<u>Thm</u> (CGGS) $\Pi_{w,u}$ is related (up to $\mathbb{C}^{\cdots} \times (\mathbb{C}^*)^{\cdots}$) to braid varieties of 4 different braids both on k strands and on n strands.

Thm (Gao, Shen, Weng) $X(\beta)$ (up to $\mathbb{C}^{\cdot\cdot} \times (\mathbb{C}^*)^{\cdot\cdot}$)

Thm (Gao, Shen, Weng) $X(\beta)$ (up to $\mathbb{C}^{\cdots} \times (\mathbb{C}^{*})^{\cdots}$) has a structure of cluster variety if $\beta\Delta^{-2}$ is a positive braid.

Problem: What does it tell us about link homology? One example next time.